Review of the Relativity: from the Special to General

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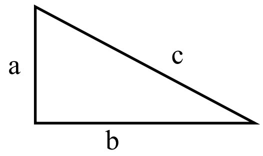
Abstract: The relativity breaks the conventional notion of time and space that established by Newton mechanics, and is widely verified in various experiments. In this work, we briefly review the basic theory of the special and general relativity. The preliminary mathematical knowledge including the Pythagorean theorem, vector calculation, and the multidimensional spacetime is introduced first, which further draws the special relativity. Then based on the theory of light cone and Euclidean space in curvilinear coordinate, we recall and complement the restricted relativity and introduce the general relativity.

1. Introduction

1.1 The Rule of Pythagoras

The Pythagorean Theorem discusses the quantitative relationship between the hypotenuse and two right-angled sides in a right triangle. Figure 1 displays the diagram of the Pythagoras Theorem. It states that the square of the hypotenuse's length is equal to the sum of the square value of two right-angled sides' length.



$$a^2 + b^2 = c^2$$

Fig.1 Diagram of the Pythagoras Theorem

1.2 Rotations and Dot Product

Rotation is a linear transformation that fixes the origin while preserving the length, which means that it is a sort of orientation that keeps the length of the origin spatially invariant when rotating. Figure 2 gives a demonstration of the vector rotation.

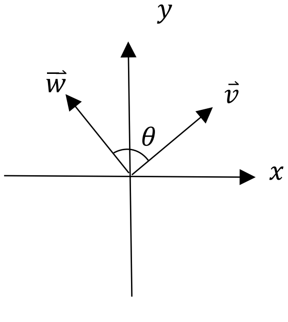


Fig.2 Diagram of the Rotation of a Vector.

In this coordinate system, there are two vectors $\vec{v}$ and $\vec{w}$. The angle between the two vectors is denoted as θ. Using this notation, according to the rule of Pythagorean, the following three formulas can be deduced

$$\vec{v}\cdot\vec{v} = {\vec{v}_x}^2 + {\vec{v}_y}^2 = |\vec{v}|^2 \tag{1}$$

$$(\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w}) = \vec{v}^2 + \vec{w}^2 + 2\vec{v}\cdot\vec{w} \tag{2}$$

$$\vec{v} \cdot \vec{w} = |\vec{v}| \cdot |\vec{w}| \cdot \cos\theta \tag{3}$$

In Newton mechanics, the displacement travelled by an object can be expressed by the accumulation of velocity dv of the object through the duration dt .

$$\mathrm{d}s = \mathrm{d}v \cdot \mathrm{d}t \tag{4}$$

Similarly, if the vector is time-dependent, the distance of the vector can be expressed by the following equation

$$distance = \int_a^b \left|\mathrm{d}(t')\right| \,\mathrm{d}t \tag{5}$$

1.3 Galilei Transformation

1.3.1 Galilei's Relativity Principle:

An event is anything that may happen in space and time, e.g., a car passing by the road at a certain point in time. Events happen at a certain point. Each event is assigned with a set of four coordinates t, x_1, x_2 and x_3 (i.e., the vector containing time and three space dimensions)

For simplicity of discussion, we symbol the three space dimensions as a vector:

$$x \stackrel{\text{def}}{=} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \tag{6}$$

1.3.2 Spacetime

When an object moves in three dimensions, as an example, an object in irregular motion on the surface of the Earth, the position $d(t')$ at each instant can be recorded as longitude vectors dx and latitude vector dy in both directions

Now we consider an object that moves in three dimensions, such as the object in irregular motion on the surface of the earth. As shown in Figure 3, the position $\mathrm{d}(t')$ at each instant can be recorded as the longitude vector dx and the latitude vector dy.

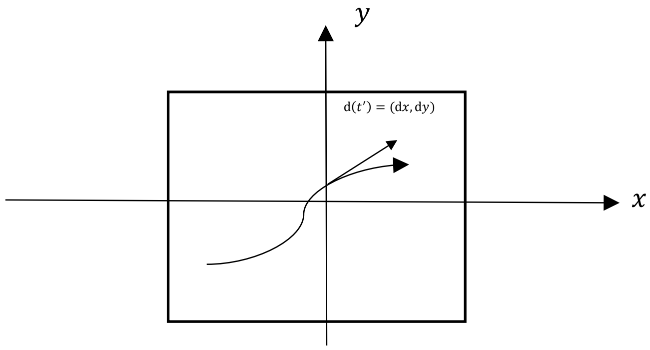


Fig.3 The Record of a Moving Object through the Coordinate

If the surface of the sphere is expanded into a plane as shown in Figure 3, then according to the Pythagorean theorem, the relationship between the move of the object's displacement and its longitude and latitude can be obtained

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \tag{7}$$

And the relationship in three-dimensional space is much more complex, where the surface of the sphere requires an additional factor to represent the displacement (the radius of the sphere is 1 unit)

$$\mathrm{d}s^2 = \cos^2 y\mathrm{d}x^2 + \mathrm{d}y^2 \tag{8}$$

This is because at the surface of the sphere, the change of dx and dy is related. And when the radius of the sphere is R, the displacement traveled by the object is expressed as

$$\mathrm{d}s^2 = R^2(\cos^2 y\mathrm{d}x^2 + \mathrm{d}y^2) \tag{9}$$

Similarly, the displacement traveled by the object can be obtained in the three-dimensional case and after adding time t

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \quad (3D) \tag{10}$$

$$\mathrm{d}s^2 = \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \quad (time\ considered) \tag{11}$$

This is the formula for an object in a rectangular coordinate system. In contrast, in special relativity, the displacement traveled by an object is expressed as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{12}$$

compared with the previous formula, a negative sign is added to the time dt^2 in special relativity, and according to the Pythagorean theorem, the vector of the object can be obtained

$$|\vec{v}| = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2} \tag{13}$$

Therefore, the spacetime in Special Relativity is expressed as

$$\pm|\vec{v}|^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{14}$$

1.3.3 The Light Cone

Light cone is the path that a flash of light, emanating from a single event (localized to a single point in space and a single moment in time) and traveling in all directions, would take through spacetime

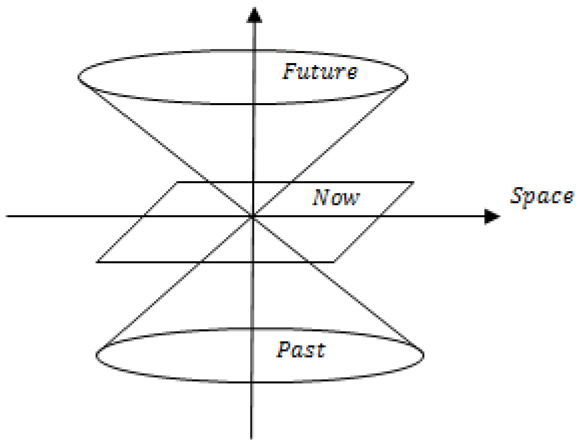


Fig.4 Diagram of the Light Cone

As shown in Figure 4, the part above the light cone is called the "Future", below it is called the "Past", and the intersection of all lines is the "Present" where a plane representing the space is marked. The vectors at the edge of the light cone are called Null Vectors (i.e., $ds^2=0$)

1.3.4 Transformation

At the end of the nineteenth century, conceiving experiments were proposed to determine the speed of the Earth relative to the resting cosmic aether. The velocity that related to the aether can only be measured by the electromagnetic effect, such as the propagation of the light wave. However, in the Michelson-Morley experiments of 1881 and 1887, no drift velocity was found, from which Einstein concluded: "The speed of light c is always constant". (Will, C. M., 2014)

Independent of the motion of the light source and the observer, light has the same velocity value in every inertial frame.

Even in three dimensions, the movement of the object in one axis would simultaneously cause the movement in the other two axes. This is also the way that Euclid's attempt to interpret the space through the plane in Euclidean Geometry. (Ludyk, G., 2013)

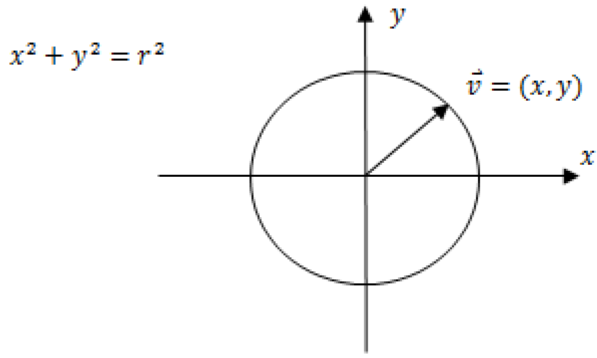


Fig.5 Vectors in Euclidean Space

For normal rotations in Euclidean Space, the vector is rotated from the origin. Although the direction is changed, the magnitude of the vector does not change, like the rotation in Figure 6, vector rotate from $\overrightarrow{v_1}$ to $\overrightarrow{v_2}$ in a circle.

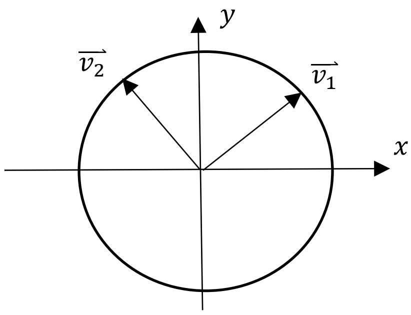


Fig.6 Normal Rotations in Euclidean Space

For Lorentz Transformation, the vector is rotated from $\overrightarrow{v_1}$ to $\overrightarrow{v_2}$ along the hyperbola of Minkowski Spacetime as shown in Figure 7.

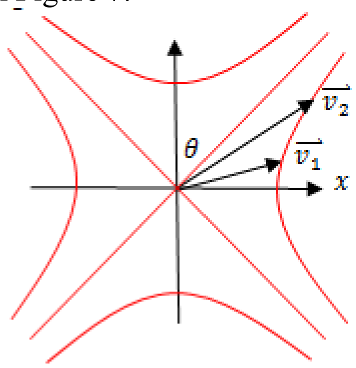


Fig.7 Boosts in Minkowski Spacetime

Rotations or Boosts preserve dot products except for reflections. There are the only linear transformations that preserve the dot product.

To sum up, the reference system transformation of Galilean spacetime is a rotation along a straight line, so the transverse time t remains constant and the time distance between events is independent of the reference system.

Minkowski spacetime's frame of reference transformation is a rotation along a hyperbola, so t^2-x^2 remains constant, and the time interval and even the direction are related to the frame of reference.

1.3.5 Special Relativity in Polar Cords

When the object is on the surface of a sphere of radius r with longitude φ and latitude θ, then at this point it can be expressed as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \mathrm{dsin}^2\theta\varphi^2) \tag{15}$$

1.4 Newton Metrix

Newton's fundamental laws are invariant also with respect to such a transformation. A general Galilei transformation is determined by 10 parameters: t_0, $x_0 \in \mathbb{R}^3$, $v \in \mathbb{R}^3$, and $D \in \mathbb{R}^{3\times3}$. The rotation matrix *D*, in fact, has only three main parameters since any general rotation consists of successive rotations performed around the x_1-, x_2- and x_3-axis, so the whole rotation is characterized by the three angles φ_1, φ_2 and φ_3, where, for example, the rotation around the x_1-axis is achieved by the matrix (Ludyk, G., 2013)

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_1 & \sin\varphi_1 \\ 0 & -\sin\varphi_1 & \cos\varphi_1 \end{pmatrix} \tag{16}$$

2. General Relativity

After the creation of special relativity, Einstein found that gravity did not satisfy Lorentzian covariance and could not be incorporated into the special relativity framework, so he created a new theory of gravity called general relativity.

When this gravitational theory was proposed, many unsolvable problems in the view of conventional theory were finely explained. For example, it perfectly explained the Mercury incoming motion. Moreover, it predicted the angle at which starlight would be deflected when passing near the Sun, which was confirmed by experimental precision.

In addition, general relativity also predicted the expansion of the universe, black holes, gravitational waves, and gravitational lensing effects, which were eventually observed one after another and fit perfectly with the theory.

2.1 Einstein Equation

General Relativity is equal to the idea that "Matter curves spacetime". In general relativity, time and space are bent, while in special relativity, though the time and space are closely related, they are not bent. And this tendency to bend makes objects attract each other, which is also the gravitational force in Newton mechanics. For general relativity, the most widely used formula is known as Einstein Equation

$$G = 8\pi T \tag{17}$$

Where *G* is known as "Einstein Curvature Tensor", and *T* represent "Stress/energy Tensor", which is "matter density".

2.1.1 Schwarzschild Spacetime

Because of its simplicity, the solution of Einstein's field equation will now be determined for the outside of a spherically symmetric, uniform, and time-invariant mass distribution. The first exact solution of Einstein's field equation was given in 1916 by Schwarzschild.

Generally, the Schwarzschild equation is written as

$$\mathrm{d}s^2 = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2) \tag{18}$$

Besides it can also be written as the following form:

$$\mathrm{d}s^2 = -\left(\frac{1-\frac{m}{2R}}{1+\frac{m}{2R}}\right)^2 \mathrm{d}t^2 + \left(1 + \frac{m}{2R}\right)^4 (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{19}$$

Where $R=\sqrt{x^2+y^2+z^2}$, and as $m=0$, degrades to $-dt^2+dx^2+dy^2+dz^2$.

2.1.2 The Relationship between "Gravity" and Spacetime in General Relativity

The concept of gravitation is not needed in general relativity, or it is datamined. Gravity is not a real force, but a curvature of spacetime. It is best summarized by Wheeler's statement: spacetime tells matter how to move, and matter tells spacetime how to bend.

The Earth moves elliptically around the Sun, not because it is subject to the Sun's gravity, but the Sun bends space-time around it.

In the language of mathematics, the above process can be described as follow:

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\frac{8\pi G}{c^4}T_{\mu\nu}\ \textit{(Matter tells spacetime how to bend)} \tag{20}$$

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}+\Gamma^\mu{}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}=0\ \textit{(Spacetime tells matter how to move)} \tag{21}$$

When there is no sun present ($T_{\mu\nu}=0$), the spacetime is flat Minkowski Space. When the earth moves along a straight line, and the equation of the line is

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}=0, x^\mu=v^\mu\tau+a^\mu \tag{22}$$

When the Sun exists ($T_{\mu\nu}\neq 0$), the space-time around the Sun is bent. The space-time geometry is

$$\mathrm{d}s^2=-\left(1-\frac{2m}{r}\right)\mathrm{d}t^2+\left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2) \tag{23}$$

When the Earth moves along a "straight line" in curved space-time, and the equation of the line is

$$\frac{D^2x^\mu}{D\tau^2}\equiv\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2}+\Gamma^\mu{}_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}=0 \tag{24}$$

If the formulae are compared when there is the sun and when there is no sun, from the results of the comparison, we can find a major feature of general relativity, that is, the laws of physics have the same form in any reference system. If the Earth moves in a straight line in flat spacetime $\frac{d^2x^\mu}{d\tau^2}=0$, then in curved spacetime, the Earth still moves in a "straight line" in curved spacetime $\frac{D^2x^\mu}{D\tau^2}=0$.

In both cases, there is no gravitational force. This confirms the description at the beginning, that general relativity does not require the concept of "gravity".

2.2 Curvature of Spacetime

Rieman curvature tensor, which is also called "Riem Curvature" to show 4×4×4×4 ($i\times j\times k\times l$, $0\leqslant i,j,k,l\leqslant 3$) matrix at every point. Thus, there are 256 numbers and 20 distinct.

Ricci curvature tensor, which is also called "Ric Curvature" to show 16 numbers and 10 distinct.

Einstein curvature tensor, the equation is shown as

$$G=(Ric)-\frac{1}{2}\cdot S\cdot g \tag{25}$$

where S is a number, and $G, S\ and\ g$ are $[4\times 4]$ matrix. And the $Div(G)=0$, like $\nabla\times\nabla_f=0$, or $\nabla\cdot(\nabla\times vector)=0$, forms a sort of mathematical identity.

For its mathematical identity, the equation can be written as

$$\nabla\cdot\left(Ric-\frac{1}{2}\cdot S\cdot g\right)=\vec{0} \tag{26}$$

More specifically, for Vacuum Einstein Equation, which is widely used in black holes, gravitational waves have the following expression

$$G=0 \tag{27}$$

$$Ric-\frac{1}{2}\cdot S\cdot g=0 \tag{28}$$

$$8\pi T = 0 \quad (29)$$

The equation deduced shows the conservation of energy and momentum.

$$0 = \nabla \cdot T \quad (30)$$

And for Hibbert-Einstein Action

$$F_{(g)} = \int u \ S_g \, \mathrm{d}V_g \quad (31)$$

where $F_{(g)}$ means the matric of spacetime, $\int u$ means to the universe and S_g is the number at each point of universe.

These three curvatures are the sources of the "gravity" phenomenon.

References

[1] Einstein, A. (1982). How I created the theory of relativity. Physics today, 35(8), 45-47.

[2] Ludyk, G. (2013). Einstein in matrix form.

[3] Will, C. M. (2014). The confrontation between general relativity and experiment. Living reviews in relativity, 17(1), 1-117.